Ciphers and Commuting Algebras of Hilbert Spaces in Music

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Abstract

The musical staff notates Pitch Value Vectors whereas tablature, using fret numbers on string lines, denotes Position Value Vectors, forming a commuting algebra of Hilbert Spaces. In 2001 I demonstrated that music is semi-algebraic (Allen and Goudessenue).

Pitch Value Space is undefined without a connection to pitch, and when connected to pitch by a barycenter, becomes defined and complete.

A defined musical system must have at least 2 functions, the chromatic f(x) and the harmonic function g(x) that form a composite function with at most 1 common center (Music Multicentricity Theorem). Thus tonality is defined by the line of tonal projection that marries pitch to position to make a musical tone.

Since musical systems must have a tone generator (instrument or device) the music topos must be the triple composite function $f \cdot g \cdot h$ where f(x) is $a + b + c = 0$ and $g(x) > 0$ is a scale center and $h(x) > 0$ an instrument center. A music cipher as defined here as an affine projection that marries R:Z pitch to position to compose a note [tone point as an orthonormal pair (position value, pitch value)]. The harmonic message is embedded in a musical system by the cipher which defines tonality, so that (harmony, tonality) is another orthonormal pair.

A cipher can also make a new note from one already known in a system.

The only algebraic operation in a musical topos is vector additions to a single barycenter according to a difference function defined by the complete lattice of the musical system, and according to the Boolean Arithmetic Operator of the Music Cipher which forms the geometry of tone value spaces by its prime ideals. The cipher model is therefore simple and natural compared to current music topology requiring two centers and several algebraic operators.

Music is composed by the finite union of notes and open intervals defined by the composite functions of the fundamental, the key, and the intonation algorithm.

Tonality, the sum total of every function, relation, and element in a musical system, is the same as the algebraic-logic interface (numeric key) of the pitch-position intonation algorithm that is precisely the triangle of cipher vectors formed between one logic and at least two algebraic sub lattices. The cipher vector defined by a complete musical lattice is also the same as the arithmetic tone values closure operator that defines tonal geometry. Specifically, the cipher is precisely the projection between the logic sub lattice and at least two algebraic sub lattices in the musical system, where the sub lattices all share the fundamental as 1 common center.

Therefore the cipher is equivalent to a point, a line, a triangle, and a sphere, reflections resulting from line-point duality in geometry. Without a common center for the R: Z cipher the musical clock is undefined: Euler's donut is dead.

The new model is a clock: the fundamental is the hour hand, the instrument position is the minute hand, and scale position is the third hand. Tonality, like time on the clock, is a vector as a composite of three functions with 1 fundamental in common. Therefore, tonality has at least two functions but at most one center.

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Cipher Definitions

1) Zero

2) Arabic numeral notation collectively

3) A secret method of writing, as by transposition or substitution of letters

4) Writing done by such a method; a coded message

5) The key to a secret method of writing

6) A combination of letters, as the initials of a name, in one design; a monogram

7) To use figures of numerals arithmetically

8) To write in or as in cipher

9) To calculate numerically; to figure

10) To convert into cipher

11) A person of no influence.

Also Cipher Text: Code Text; the encoded version of a message; cryptogram. Cipher text is the opposite of plaintext.

Adapted from Random House Dictionary of the English Language Second Edition Unabridged 1987

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Chapter 1 INTRODUCTION

Section 1.1 Growing the Music Topos: Codes and Composite Functions

Musical compositions and codes are the same because they are commuting algebras where one element is substituted for another by addition. Both music and codes are composed by adding R:Z vectors I call ciphers to make a topos that is a vector field formed by a Boolean Arithmetic of prime ideals to make musical systems and caesarian codes that are composed only by the finite union of points and intervals formed about a single barycenter.

Every possible structure in music is formed by a single R: Z cipher vector that is an algorithm that marries pitch in R to tonal values (pitch values and position values) in Z by a barycenter. Equivalently, the cipher is a vector, a difference function between any 2 tone value vectors.

This theory is based on a heuristic method by a developed by musician and mathematician who began making rules on guitar where an algorithm seemed impractical, and directly observed orthonormality of music triangles, leading eventually to discovering a new general form of the musical topos M=(x, {f(x)}, {g(x)}) in which x is the fundamental, f is the Chromatic function x = x + 1 (musical scale function) and g defines the tonal center of the key, which I call Special Tonality since concert pitch is fixed, or more generally in a tuning space as M=(x, {f(x)}, {g(x)}, {h(x)}). This topos expression defines the clock theory of tonality where the fundamental is like the hour hand, the instrument position is like the minute hand, the key is like the second hand, and the numbers on the clock are the discrete tone position values.

After years of study I am now able to leave the empiricism of the guitar arithmetic behind and formulate a total new and different theory of tonality from the arithmetic and geometry that I learned on guitar from which I make a natural, defined but not always smooth topology.

Musical topology is traced to Euler who first described pitch value space as Euclidian using a planar pitch value triangle where the distance metric is $a^2 = b^2 + c^2$, when in fact the correct metric is $a = a^2$. Furthermore, with 2 centers, Euler’s donut is not simple, defined, nor complete.

After almost 300 years of fallacy, a guitarist kills Euler's donut, perhaps the only egg he laid, by following Euler's own binary path to a new level of understanding, where once again the Kuhnian center of the music world is not where we thought. Music topology is semi-algebraic and complete, and this discovery has important technical and aesthetic implications.

I am assuming that the reader is familiar with semi-algebraic theory, and in particular with the formulation by van den Dries. Dual tonality is the result of introducing semi-algebraic theory to music.

Section 1.2 Tone Values Point Sets to Musical Scales

To grow a musical topology we begin with tone value point sets {1, 2, 3 , …} = {A, A#, B, C …}. A Space constructed from these elements alone is a Cantorian Hell (Marker 2010) because of Gödel's Theorem.

Any 2 tone values sets are disjunct (that is, independent/orthogonal) and by theorem the must be 1 (and only 1) function relating any 2 sets in disjunction. (See Appendix (?#) Mathematical Theorems Used in This Study)

The tone values point set is then tested for categoriticity, compactness, and countability, and then is mapped onto the real closed field defined by 1 octave that is created by a factor of 2 that defines precisely the open interval between 0 and 1. This leads finally to the Tarski-Vaught Test for a complete lattice.

Once the musical lattice is shown to be defined and complete, it follows directly that tonality is the algebraic closure operator of the lattice defined by the vector between sub lattices so that the topology and the R:Z cipher vector are the same.

I am skipping over the way that the open interval is defined as a Cauchy because while this is critically-important this is also well-known. Briefly, however, the scale sequence is S = $\{s_n\}$, n ε 12 is a sequence in a metric space X, D such that s_n □ y and s_n □ y', then y = y'. That is, a sequence in a metric space can converge to at most 1 limit (Gemignani 1967).

The tone value point is defined in the lattice by the greatest lower boundary and least upper boundary. This means every tone path is the union of an open unit interval (1/12 of an octave) and 1 point.

Therefore music is the union of notes and intervals that are added to a barycenter like a code using a pitch-position algorithm and not a function.

This mathematic theorem stipulating at most 1 point defines tonality has profound implications because it establishes an important requirement for a simple topology: Music topology cannot, by theorem, be based on more than 1 point of origin.

Section 1.3 A Simple, Natural Topos is Defined a Cipher Vector

In the subsequence discussion I take it as given that a simple, natural, completely-defined music topology is established by a single R:Z vector between the continuously-increasing monotonic function of pitch and the tone values point set so that the properties inherent in the real number line are extended through a single barycenter to the tone values vector stack.

Section 1.4 A Pitch Space or Pitch Class Space Is Absurd

A pitch space, and a pitch class space, in which pitch values are already embedded in the real closed field without a cipher is undefined and pure nonsense. Furthermore even if pitch value space could exist without a cipher, the logic sub lattice would still imply orthonormality.

Section 1.5 Ciphers: The Elemental Tonal Function: X = X + 1 or Y=X+B?

Chromatic functions, originally a spectral reference to chroma (color), is a commutation of tone values by accidentals (sharps and flats), where tonal movement occurs only by 1 step at a time (semitone).

Harmonic functions, defined by a secondary tonal center independent of the barycenter (that is to say, orthogonal to the fundamental), as the harmonic function $g(x) > 0$ where x identifies the musical key.

Like the system fundamental (the barycenter), the musical key can have only 1 center. This follows since a partially ordered set can have only 1 less than function.

In the harmonic function, the interval been 2 notes can assume any value allowed in the system but the distance between any 2 notes, regardless of the tonal interval, is always 1.

This observation flies in the face of traditional music theory which claims that the harmonic path in music is only by 5 or 7 steps since these are the only paths in which every pitch value class is included (great circles).

Section 1.6 Musical Graph Theory: Every Distance is 1 for Both Chromatic and Harmonic Functions

In fact the correct meaning of $g(x) > 0$ is that movement between any 2 tones is equivalent and always a distance of 1. Stone in 1939 showed that this universal metric of 1 is a property of all logic sub modules. This principle is used in circuit logic diagrams, which like musical graphs, can always be drawn on paper in a useful way (Hasse Diagrams) but can never be drawn correctly (K3, 3 subsets or utility graph problem).

In the music topos we can expand the Theorem of Hass Diagrams to say we can always draw musical graphs in a useful way, but we can never draw them accurately and we cannot understand why the graphs are not accurate by ear any more than we can see they are not accurate by looking.

This universal distance metric where every musical object is a distance of 1 from every other musical object is not intuitive and has apparently been over-looked since it seems to defy common sense.

In music any 2 notes, scales, chords, keys, tunings and any other set of tone value elements are always located in pitch-position space at a distance of 1, regardless of the number of chromatic steps. The difference between distance and interval is inherent to semi-algebraic number rings which exist in Hilbert Spaces.

Chapter 2 THE DISCOVERY OF TONE VALUE CIPHERS

Section 2.1 Guitar Tuning Ciphers

I first discovered ciphers as arithmetic operators that change the tuning and key of guitar music.

The first cipher I used changed guitar music in Standard Tuning to Drop D Tuning. I made the cipher as a difference function between the open string tuning notes EADGBE and DADGBE. I used the word cipher in standard cryptologic terminology: an algorithm formed by 2 code algorithms and used to transform one code to another.

I was surprised that there were 2 ciphers that could transform Standard to Drop D Tuning, and I noticed that the tuning cipher and key cipher could be added to make another cipher.

I had then discovered my first example of the pitch-position triangle as tuning vectors and I saw that the tuning cipher, the key cipher, and the tuning-key cipher formed an equilateral right triangle on the surface of a spherical manifold.

This created a paradox since in traditional music theory musical topology is a planar or toroidal manifold in Euclidian Space.

Clearly music cannot be Euclidian and Non-Euclidian at the same time because a plane and a sphere are not homeomorphic.

But the point, the line, the equilateral triangle, the logic sub module, the tone value point sets are all homeomorphic. Therefore we may choose to think of tonality as a point, a line, a triangle, a clock, a ball, and a sphere but if we think music is a plane or a donut, then we are unable to see that tonality and harmony are distinct because the system fundamental and tone value key center have no defined relation.

Section 2.2 Tracing the Origin of Orthonormality

When I first realized that the pitch-position slope is always 1, I thought orthogonality was due to tempering the scale, but this is not correct.

I observed first that the guitar tone value vectors were always orthogonal and that any 2 tuning-key states could be transformed by 1 cipher vector operator with algebraic and logic functions.

I realized that the pitch-position triangle is always orthonormal (that is, an equilateral right triangle). The rule $a = a^2$ is the law by which guitar fret are constructed geometrically, so that the conclusion that music is a Hilbert Space is in escapable but also easily over-looked since like on the surface of the earth we merely follow the musical sequence and are not required to think about what space we are in.

Musical ciphers are interesting because the cipher itself creates tonal duality (well, really a tonal multiplicity) by the manner of embedding the harmonic image. Duality (in its most elemental form) results from the fact that pitch is a projectile invariant because that is how we identify tones, while instrument positions are not directly audible and scale position requires some expertise to identify.

Music ciphers can collineate pitch (that is they cipher or zero out the pitch leaving the value unchanged) and commute position, or they can collineate position and commute pitch. Ciphers are therefore the arithmetic operators for tone values in music.

Section 2.3 Tonal Redundancy and Duality Principle

Tonality, short for tonal expression, is the sum total of every possible function, relation, and tone value elements in a musical system. It is surprising then that so many theories of tonality have been presented (Tymoczko 2011).

Tonality is most closely associated with the musical key, but if musical keys are parallel and co-incident structures then how can tonality vary with key when keys are all the same?

Tonal duality is the result of the marriage of pitch to position to make a musical tone that is an orthonormal pair. Collineation means tone value can be encoded in a musical system by pitch, or by position, and by a composite function of both pitch and position.

A tone encoded by commuting only pitch or position value is said to be "ciphered" or zeroed out: In particular pitch values can be encoded (collineated) strictly by position, which we cannot hear. The pitch values cipher out to zero (but the position values commute). This cipher-commutation interaction is the fundamental *pitch-position duality principle*:

(pitch value (1), position value (1))↔ (pitch value (1), position value (2))
[Commuting Position, Ciphering Pitch]
(pitch value (1), position value (1))↔ (pitch value (2), position value (1))
[Commuting Pitch, Ciphering Position]
(pitch value (1), position value (1))↔ (pitch value (2), position value (1))
[Commuting Pitch and Position Together as a composite]

The duality principle means 1 pitch can be found at 2 positions (Position Redundancy) and 2 pitches can be found at 1 position (Pitch Redundancy) in different tuning states.

Section 2.4 Musical Lattice

It is a trivial exercise in lattice theory to show the musical lattice is complete. I am also skipping over transitive, associative, and distributive properties of the lattice for brevity.

An important principle that whatever is true for pitch values must also be true for scale and instrument position values: ([Tone Values Symmetry]).

Every musical lattice has 1 and only 1 barycenter.

Section 2.5 General Pitch-Position Relation on Guitar: Σv = Prime Ideals

The Summation Vector Σv = Prime Ideals defines the determinative tuning summation vector which gives the open string tuning notes on guitar that are the prime ideals of the guitar fret board matrix tone value field, and is the same as the fundamental tones of the individual strings that form a tuning by their tempered union to create a tempered pitch-position space. The pitch-position space is defined by the tuning algorithm vector. The 1 x 6 tuning algorithm vector and 6 x 1 summation vector on guitar are seen to be perpendicular.

Standard Tuning Vector V=(0, 5, 5, 5, 4, 5) defines Σ v=(0, 5, 10, 15, 19, 24).

If barycenter is defined as Pitch Value 0 = E2 then Σ v for Standard Tuning is evaluated as EADGBE (short notation for E2 A2 D3 G3 B3 E4). Any other note on guitar is formed by adding to these 6 open string tuning notes that act as a tertiary tonal centers (after the fundamental and key centers). This is my first orthonormal vector pairs (tuning algorithm vector, Σ v) and (Σv, Cipher).

Note that on any guitar string the fret number is the position value and the open string note plus the fret number is the pitch value.

Section 2.6 Position Redundancy

Tonal redundancy (the basis of duality principle) is shown in equation (1) which I discovered on guitar but is generally true for any tuning space also:

pitch value vector = position value vector + Σv

[Equation 1 General Relation of Pitch and Position in Tuning]

I assert that this equation describes the general relation of pitch and position vectors in a tuning space as a commuting algebra of Hilbert Spaces.

Chapter 3 THE FUNDAMENTAL IS A PITCH-POSITION TONE, NOT A PITCH VALUE

Section 3.1 Fundamental Tone Defined

The fundamental is not defined by either a pitch or a position value, but instead by the marriage of both pitch and position by the intonation algorithm cipher.

The fundamental is not just a pitch but rather an orthogonal pair (zero fret position value on the string, 0 pitch value for the string), that is the same as the fundamental the cipher note that defines the harmonic function g(x)>0 in the partially ordered sequence of the scale 12-arity.

Tuning means the adjustment of a fundamental pitch value at a zero string position, a position defined by string mass, tension, and length.

A tuning therefore is always an R: Z union of discrete and continuous variables, the string number (a position value) and the monotonic frequency function. It is fallacy to think the fundamental is the same as the frequency itself since clearly the pitch value can occur without being a system fundamental.

Every musical tone, of course, is defined by its own fundamental, which is the lowest frequency of vibration, but the question here is whether or not the fundamental is primary (known only by the system) or secondary (formed by addition to known values).

Section 3.2 Fundamentals and Tunings

A musical system can have only 1 fundamental tone since there can be at most 1 lowest tone in any system.

Tuning in a music topos is defined by N strings (tone generators) which are tertiary tonal centers that define the numeric key of the tuning algorithm (that is the tertiary tonal centers that are not the tonal center of the observed musical key) known by their tonal relation to the tuning barycenter (the "center of gravity" about which tonal movement is understood as a function of the system fundamental).

The summation vector Σ v, as noted, is constructed from the Tuning Algorithm Vector. The tuning vector gives the tone values interval between strings barycentrically and Σ v gives the fundamentals of the strings in barycentric coordinates that are the prime ideals.

A zero in the first vector position of a tuning vector or summation vector is a place holder for the barycenter = system fundamental. This is called vector expansion.

Section 3.3 Theorem: Music is a Hilbert Space

I prove Hilbert Space in music:

We are given a 12-tone tempered scale point set (tempering is not essential). In a musical graph we have on the x-axis pitch values and on the y-axis position values (which may be scale position, fret number, piano key number, other pitch values sets and so on). Note that tone value sets are disjunct sets. The pitch-position triangle is constructed on (1, 1) where 1 is defined by 1 octave such that (0, 1) is the real closed field. We have then a K12 number ring mapped to the real closed field (0, 1). Therefore the pitch-position triangle has 3 sides equal to 1, and 3 right angles. The path of tonal movement around the triangle is binary where each vertex index is 2. There are 12 steps on each side of the triangle and the real closed field ensures orthogonality as well as orthonormality.

Let $a = 1$. Therefore $a = a^2$.

Since every pitch-position triangle satisfies the condition $a = a^2$, it follows that any space constructed as a series of pitch-position triangles must be a Hilbert Space.

Next, using the arithmetic of the guitar tone values vectors I show that the inner product (actually sum) for tone values vectors is orthonormal by simple examples.

Section 3.4 Cipher Arithmetic Examples

The Summation Vector (Σ v) is the binary-algebraic interface (also called the numeric key) of the cipher algorithm, which is based on the cipher as a difference function:

cipher vector = Σ v1 – Σ v2 [Equation 2 Cipher Vector Expression]

By convention, I orient the Tuning Algorithm Vector as the 1 x 6 vector, and summation vector Σ v as the 6 x 1 vector. These vectors are clearly perpendicular visually, logically, and algebraically.

The tuning algorithm vector and the summation vector are therefore the first orthogonal vectors I observed, and also normalized on the real closed field (0, 1). Therefore every cipher has length 1 and is orthogonal to tuning vectors and also other ciphers.

Tonal Arithmetic Example 1: Standard Tuning To Drop D Tuning Ciphers

We have Standard Tuning Vector V1 = (0 5 5 5 4 5) that gives Standard Tuning Summation Vector Σ v1 = (0 5 10 15 19 24) which is usually interpreted to mean EADGBE. (Incidentally, there is no sense in which Standard Tuning is a standard any more than English is the Standard Language of the World.)

Note the tuning algorithm is independent of the barycenter.

Drop D Tuning is the same as Standard except that String 1 is lowered 2 steps; Drop D Tuning Algorithm Vector V2 = (0 7 5 5 4 5) and Drop D Tuning Summation Vector is Σ v2 = (0 7 12 19 21 26) usually known as DADGBE.

The tonal arithmetic for the Standard to Drop D Tuning Cipher is calculated Σ v1 – Σ v2 = (0 5 10 15 19 24) – (0 7 12 19 21 26) = (0 -2 -2 -2 -2 -2) per equation (2).

Section 3.4 Orthogonal Tuning-key Cipher Triangles

(Tuning, Key) is another orthonormal pair.

A tuning-key cipher is formed by adding together a key cipher and a tuning cipher to make a triangle:

tuning-key cipher = tuning cipher + key cipher

[Equation 3 Tuning-Key Cipher Expression]

Section 3.5 Distance Metric Makes Tonal Values Space a Topology

Equation 3 is the cipher distance formula equivalent to the Pythagorean theorem except instead of a distance metric $a^2 = b^2 + c^2$ the triangle is described by the distance metric $a = b + c$ which is $1 + 1 = 1$ (because $a = a^2$).

The ciphers length is 1 since every element satisfies $a \equiv a^2$. The cipher then is the point, the line, the triangle, and the sphere at once. A pitch-position relation then is a point, a line, a triangle, and a sphere depending on barycentric perspective.

Tonal Arithmetic Example 2: Using the Standard-Drop D Cipher in the example above is decomposed into an equilateral right triangle.

Arithmetic: (0 -2 -2 -2 -2 -2) = (2 0 0 0 0 0) + (-2 -2 -2 -2 -2 -2) per Equation (2).

This equation shows that the inner product (sum) of ciphers is zero.

This is an equilateral right triangle where the vertices are Standard Tuning Key of E, Drop D Tuning Key of D and Drop D Tuning. The 3 ciphers are equal in length and their vertices locate 3 tuning-key states in tuning space. They are located on the surface of a spherical manifold, where all tuning-keys possible are a distance of 1 cipher form the center.

The first cipher in the arithmetic above lowers key and changes tuning both at once, the second cipher changes tuning only, and the third cipher changes key only. Tuning and key ciphers add, and I will show these are non-rotational Abelians.

The example shows exactly how I first observed the tuning-key cipher and understood it is made from two summation vectors and therefore is a true cipher made from two tuning algorithms. It is a very simple example but clearly shows a commuting algebra of Hilbert Space.

Section 3.6 Cipher Logic Functions

The following examples show the vector pair (algebra, logic) and their composite functions.

Tonal Arithmetic Example 3: Tonal Logic is also a form of tonal values addition (logic cipher functions)

After tone algebra on guitar, to change tuning and key you move the notes at least once more, and commonly several times in proofing.

Logical tone movements are required in commutations to ensure the best position. The logic functions are functions of the tuning algorithm vector defined by the system fundamentals. Not all notes move in the logic operation step (notes may already be at best position) and notes with redundancy 1 do not move logically at all because they have nowhere to go.

The logic vector is an important source of expressivity; its effects are not foreseeable and not directly audible by pitch value but only perceived in performance. This creates a rich syntax and semantic structure, as well as an illusion of the highest ordered create by a secret cipher code, since Σ v1 requires a certain path of tonal movement and Σ v2 requires a completely different path of movement even on strings that are not even altered algebraically by the cipher operator.

The listener cannot possible imagine the path of tonal movement on guitar (in 6 dimensions we can't hear!) but the tonal expression of the harmonic image none-the-less affected by discontinuity and symbol redundancy [Duality Principle].

Section 3.7 "Scordatura Changes Everything"

If the Σ v is altered in any way, the musician will find that it is impossible to play without relearning the entire fret board pattern. Mastery of (0 5 5 5 4 5) tuning vector algorithm does not confer knowledge of (0 7 5 5 4 5). On violin, similarly, scordatura changes everything. Scordatura, like tablature is a position vector notation under Equation 1.

There are special positional notations for scordatura that do not require re-learning pitch-position relationships (which is very difficult), however this method does not work well because without knowing tuning logic the syntax and semantics are not correctly understood so the effect is rather like translation by a non-fluent speaker using a dictionary. The pitch values are commuted to new positions but the correct position is not understood.

The musical staff is a pitch value vector notation which does not change with tuning. Tablature staff and musical staff are therefore orthogonal staves related by Equation 1. By itself the music staff does not define guitar music and traditionally additional notations are required for the guitarist to read the music. This problem is easily understood by Equation 1.

Chapter 4 EMBEDDING THE HARMONIC IMAGE: (MESSAGE, STRUCTURE)

Section 4.1 Expressivity and Symbol Redundancy

A binary language has no expressivity and no structure, just information.

Importantly, meaning and order is lost in a binary language when a single binary value is lost.

But in music and spoken language as well we can still understand the meaning of the message when even as much a half the symbols go missing.

On guitar this problem is very evidential since the musical keys are not parallel, co-incident structure. If the Key of G Standard Tuning is the same as the Key of G Open G, why do the 2 tunings have completely different tonality (like night and day), since the tunings have essentially the same notes?

The answer is that the tuning determines the key and therefore the tuning is the real determinant of tonality, while the key defines harmony which is the expression of pitch values in isolation from position values.

We shall see that the principle "tuning defines tonality" is evident in a single string.

Every Σ v invokes its own set of tone logic rules which the guitarist must follow to render the harmonic image. This means that the harmonic message written in pitch values is married to position values by the Summation Vector Σ v, and the message is the harmonic expression while tonality is the expression of position values in tuning structure.

The line of tonal projection is the medium in which the harmonic image is embedded.

Section 4.2 Arithmetic Example 4: Cipher Semantics and Syntax

E chord in Standard is the position vector (0 2 2 1 0 0). The Standard to Drop D Cipher in Example 2 transforms the E chords as calculated using Equation 1 as (0 2 2 1 0 0) + (2 0 0 0 0 0 0) = (2 2 2 1 0 0). The new E chord in Drop D is (2 2 2 1 0 0) which is logically valid, but will be initially rejected as semantically-incorrect by a Standard Tuning-Only Player.

The Drop D player will find there is considerable semantic position value in the half-bar formed by this E chord that does not occur as a geometric option in Standard Tuning, and anyway there is no other first position E chord available to substitute (an example non-redundant position vector; position vectors can also be impossible to play for several reasons).

Arithmetic: The position vector (0 2 2 1 0 0) in Standard Tuning is pitch value vector (0 7 12 16 19 24) and in Drop D would be pitch vector (2 9 14 18 21 26) (same key) or (0 7 12 19 24 ?) (key drops two steps by adding key cipher, the symbol "?" means that there is no replace note for String 6 in the new tuning so the guitarist will generally use (6, 4) as an enharmonic replacement (another note in the chord) for lost note on String 6.

Note the Drop D tuning chord is similar to and equal to the E chord but not co-incident.

The chord voicing in Drop D has changed, and therefore the harmonic expression has changed. The change in tonality between Standard and Drop D Tuning is distinct but also can be hidden by the guitarist as a musical trick that allows the guitarist to suddenly make a completely unexpected tonal movement the listener cannot possible understand.

The cipher results are therefore merely congruent (similar and equal), but not equivalent, which is an independent duality between orthogonal elements in the pair (harmony, tonality).

The example of the first two vectors (E Chords in different tunings) sound alike but function very differently in expression, while the second chord in D is very different in both sound and function. The D chord is voiced differently harmonically. A single position vector results in three very different chords with the same harmonic image but different tonality under different Σ v vectors. These examples show how the cipher works on guitar where I discovered equations 1-3 heuristically.

Section 4.3 Cipher Arithmetic Equations: Conclusions

My cipher equations show that any set of tone values can be changed to any states of musical system by one cipher where the cipher algebraic function is in the general form for any binary path of tonal movement (that is a binary walk of note and interval) that any cipher must follow. The cipher path must be Y=X+B where X and Y are any two tone values (not the same) and B is a vector of scalar values which the cipher adds to in order to convert 1 tone to another.

Chapter 5 REASONING OUT THE ORIGIN OF TONE LOGIC

Section 5.1 Extracting an Iso-morphic Image of Tonal Arithmetic

I noticed empirically that tone logic always followed tone algebra. The tone logic is also a form of addition. The cipher, I reasoned must be a true algorithm that is made from two algorithms! The tuning vector was the numeric key that formed the binary-algebraic interface like the notched loom card, and until now the loom was considered the earliest programmable device. But the musical string is an older algorithm by at least 2000 years! This gave me an idea which I knew was correct, and although I could not at first see why, understood that it was a powerful idea. Later, understanding ring theory better (Stone, Tarski), my initial assertion made perfect sense; indeed it is the only possible explanation of the anomalies which I observed:

Algorithm = Algebra + Logic; or more precisely Arithmetic Closure Operator = (y +x +c) + logic term ={+,>,0,1}| 12-arity

I saw empirically the algebraic functions were noncommutative, and that the Abelian group was torsion free from the lack of rotation in pitch-position graphs. My early understanding was that algebraic closure simply acted to confined tonal movement to an octave; later I realized the closure operator is a complex set of functions that make a highly sophisticated tone language, so that the cipher by itself has vector-defined cultural value function, which is the same type of value that an expressive language has over binary code. Without paired logic functions (redundancy, playability) the (originality, expressivity) would cipher out.

Section 5.2 Aesthetic Significance of Harmonic Distortion: Flatlanders Meet Jimi Hendrix and the Beatles

The diverse tonality of popular cannot be created in 1 tuning. When the harmonic image of the tonal arithmetic closure operator (that means the guitar tuning) has a pleasing discontinuity, the discontinuity can be particularly expressive in a harmonically useful way. The cultural value (originality, expressivity) orthonormal of the tuning can be an extra-ordinary phenomenon (See Box II Origin of Heebeegeebee).

Guitar tunings are far from all having the same tonality, but the listener like a flatlander has no way to penetrate the veil of tone position ciphering and commutation.

The notes seem to turn and jump just like when the sphere came through the plane and tried to tell the listener there are some interesting shapes out there in another dimension.

Open G and Open D Tunings are examples of highly significant tuning structures, while the tuning EBEBEB is, judging by popular usage, not that expressive.

Severe harmonic distortions might be useful sometimes, and composers use odd tunings for effect. Similarly Lute Tuning (0 5 5 5 5 5) is smooth topologically but virtually useless in popular music.

Standard Tuning (0 5 5 5 4 5) is apparently crowned the queen of tunings, but in no way a real "Standard."

Perhaps Standard Tuning is popular because it has the least *melodic discontinuity* of any common tuning, which gives (0 5 5 5 4 5) arguable melodic superiority or perhaps just easier to learn than tunings where scales have gaps and overlaps.

But Standard Tuning is not more expressive than other tunings, and also Standard Tuning is clearly harmonic inferior to other tunings in certain dominant, major, and minor forms of tonality.

Open G is harmonically superior for blues and rock because it sounds a 6 string major chord with zero fingers. It also has a very dominant and more pentatonic tonality than Standard Tuning. Notice that (melody, harmony) are another orthogonal pair of tone values. Playability is zero for open string notes (secondary tonal centers are system ideals), 1 for every Pitch Value not in Σ v. Playability equals the number of fingers used to play a note.

Section 5.3 Ciphering Out Tone Value Elements

The zeroing by a cipher means that tonal value changes but at least 1 element remains the same in a commutative transformation.

Since we cannot hear ciphered out pitch values (that should sound the same in every tuning, at least that is what we think if notes are known just by pitch); and also because Σ v is not easily recognized to the untrained ear (if at all), the cipher embeds the harmonic message in a secret manifold of tone value space and the tonality is the image of the message in the structure of the tone value space. If the continuity of the Σ v-defined topology is relatively smooth then we are not generally aware aurally when a guitar tuning changes.

Section 5.4 Pitch Values Collineation

The tonal element basis of tonal duality by ciphers is harmonic collineation.

The usual problem on guitar is that the harmonic value of the notes is how we identify the notes but expression is at least equally affected by position. Ciphers, then zero out pitch values, so as a result we may think that music is played in the tuning we happen to be using, and later we out find this is not true [Clapton's Law].

The binary path of tonal movement is 3-fold Path with S03 Symmetry

Using the well know equation for pitch values $p = 69 + 12 \log (f/440)$ where p is pitch and f frequency, we see clearly that pitch resolves into 3 orthogonal movements.

$$P = 69 + 12 \log f - 12 \log 440$$

This is a linear equation in the form $f(x) = a + 12b + 12c$ with universal slope $12/12 = 1$ and passing through y-intercept a.

This equation clearly show the binary path of tonal movement is 3-fold if concert pitch is not fixed at 440. Note the center point of concert pitch appears in the equation twice. This equation is not in its most simple form.

Since 69 is the pitch value of 440 and in the Cauchy Series Σ 12 = 1 so 12 can be replace dwith 1, we can substitute $69 = \log 440$ so that we have the fundamental equation for the Cauchy Series as another version of $a = a^2$:

$$P = \log F \quad \text{or } P^2 = F$$

The observed 3-fold SO3 symmetry of tonal movement (See Appendix 3) that results because tones are pitch-position pairs is intriguing and leads me to the idea of tonality as a composition function so that another set of rules other than harmony existed.

Perhaps I thought, this explains why Jimi Hendrix was culturally significant as a composer: he knew another set of music rules, that were equal and similar but completely different from music theory and based instead in position logic rules.

Transposition (a word with a special musical meaning different from mathematic meaning) is literally a cipher that changes position, but could also equally-well mean to change of pitch, or pitch and position can change together as a composite function (SO3 or spherical symmetry). Notice that position values can be scale values (defined by key), and also instrument position. This compounds my argument: where the third, fourth, fifth, and sixth tonal center for guitar go if not to the same barycenter? The complete music lattice is defined by a lattice at most 1 R: Z barycenter. Transposing Piano changes instrument positions with a lever. This is a direct demonstration of 3-fold tonal movement, as well as the use of a loom-like mechanical arithmetic operator with a numeric key (lever). Notable about SO3 symmetry: two directions change pitch but another direction movement occurs without changing pitch. This is important.

Section 5.6 Music Monocentricity Theorem

I postulate here in order to explain my observations on guitar that music is like a clock that can actually have multiple centers, and must in fact have at least two centers for tones to be defined but at most 1 system center that is the fundamental tone algorithm. [Terry Allen's Music Monocentricity Theorem]

Music Monocentricity Theorem creates a fundamental duality in all music since in any defined musical system there is a system fundamental and at least 1 more secondary tonal center (in a tuning at least 3)such as a musical key and tuning centers.

It follow then that the complete musical lattice has 1 logic sub lattice and at least 2 algebraic sub lattices, 1 for each disjunct position set.

Section 5.7 Prime Ideals Are the System Fundamentals

The prime ideals of the system are then the same as the secondary tonal centers, and the same as the tuning algorithm vector (Stone, Tarski – these are reference sources not an Abelian pair).

The system fundamental (guitar barycenter) is not required to define the tuning-key (tuning-key is an object inside tuning space) but is required to match the tuning to the observed key, which in turn defines the lowest note in the system.

In other words, if I have the tuning I can find the barycenter by running the pattern up and down the fret board matrix to find a match, and that will tell me the lowest note on guitar. Capos and detuning confound this problem but trivially.

More orthonormal pairs: Capo and Detuning -- they cipher out. Detune 1 step down and capo up 1 step and the listener will perceive no change and the guitarist will see only a shortening of strings but no commuting of pitch-position relations.

Tablature and Music Staff are cryptologically independent and related by Equation 1.

Chapter 6 THEORY OF SPECIAL TONALITY: THE TONALITY OF A SINGLE CENTER

Section 6.1 Euler Failed To Recognize Hilbert Space

When the cipher R:Z connects pitch and the Z vector stack barycenter, the metricized but undefined tone values stack becomes both complete and defined but not necessary continuous.

The Hilbert Space is important then because the commuting of pitch and position is precisely the origin of Möbius' term “equal and similar but not co-incident” projection on which he based his barycentric calculus.

The principle applied here was stated by Mobius “Every anharmonic ratio formed in a plane net is rational and depends only upon the construction of the straight lines and not upon the four fundamental points.” This means that tonality is the line of projection, not the not the pitch values.

Möbius continues, “It seems remarkable that solid figures can have equality and similarity without have co-incidence … the reason may be looked for in this, that beyond the solid space of three dimensions there is no other, none of four dimensions. If there were no solid space, but all space relations were contained in a single plane, then it would be evn as little possible to bring into co-incidence to equal and similar triangles in which corresponding vertices lie in opposite order. … If the directions of AB and A’B’ are opposite, then there in no way can a co-incidence of corresponding points be brought about by a movement of one system along a line.

Möbius describes here the problem of transposing guitar keys, and also the paradox in which tonality is a function of musical key and musical keys are co-incident so how can tonality vary?

Hint at how this problem is resolve is to note that the piano tuning is a semi-algebraic partition that separates the piano into a 2-class system of ideals where black and white keys are like odd and even numbers: either class can be considered primes.

If music compositions are all co-incident after the barycenter is removed, then music the vector pair (originality, expressivity) ciphers out to give a tone value of zero cultural significance.

Euler's mistake was while metricized pitch value space in Z when he learned about tempering and realized the tone interval was normalized, he failed to note that a pitch space is undefined without an R:Z cipher vector, and that tone value space is already metric without tempering, but still undefined without barycenter.

Pitch Value Space makes no sense. No connection to position values, no fundamental tone, no simplicity, no categoricity. Just a tonal vector stack already married to the real line, but with no single barycenter by which good things flow to the tone values. The (continuous, discrete) Abelians are undefined.

It does little to try to repair the broken donut using another vector to connect the harmonic and chromatic function since that in effect convert the torus to a sphere.

Section 6.2 Tone Value Clock Paradox

Clearly Euler didn't make a sound donut but laid a broken egg instead.

How can a musical system have two points of origin of equal weight? A pair of equal fundamentals? Two lowest notes that are not the same? A clock where the hands have no defined relation, no common center, no common 12 o'clock position?

If so then time (tonality) is undefined. We're lost. Clearly there is in music both a chromatic path and a harmonic path; but can two paths have 2 different key centers to make a donut? No. Impossible. The one pair we can't accept here is orthogonal (center, center) points.

The fundamental is unique. A fundamental is a tone, not a pitch value since the fundamental is the union formed by the marriage of the pitch value and the string. A system cannot have more than 1 lowest tone. The clock circles must have the same center, or a defined relationship.

Section 6.3 2-Fold Path of Tonal Movement

Music (be it spherical or toroidal) is always a binary function because of the orthogonal pair (note, interval).

When the movement of pitch is restricted to a 2-fold path by concert pitch, then what I call a “Special Tonality” is created. The effect is a planar projection. The 2 fold path seems to indicate independent centers but logic requires they are homotopic (have 1 center). Graph theory tells we can always diagram partially ordered sets like scales; but we can never draw them accurately. Angles and distances in dim = 2 are meaningless since they cannot be graphed accurately. That another orthonormal pair: (Tonal angles, Tonal distances).

In Special Tonality there are no tone angles because there is only 1 tonality, the same as harmony. Tonal angles come with higher dimensions.

Section 6.4 The Special Tonality Plane Converts to Donut Under Arithmetic Closure Operator

If music is planar, then under algebraic closure by the octave interval, the plane becomes equivalent to a donut. Special Tonality seems as if the key and fundamental are 2-dimension since concert pitch is fixed. But pitch-position cannot be planar in symmetry. Tonality is like the surface of the earth. A map can be used in 2 dimensions for calculations but 3 directions really exist.

Section 6.5 The Tonnetz

Special Tonality is the simplified paradigm all music students learn but doesn't work well on guitar, judging by popular use. Guitarist usually don’t read music staff, many don’t know music theory, and some very good guitarist cannot even say what pitch values they play. The intellectual standard for guitar music published today is poor and does not have the accuracy and precision found in piano music. Adding guitar chord charts is merely a memory and has not relation to chords actually played.

The tonnetz, at least several hundred years old, and has never been critically revisited and never updated. Apparently Euler realized that tempering created a metric space before topology was invented but did not properly determine the distance formula or understand the 3-fold symmetry of a union of points and lines in orthogonal Hilbert Space. He got the binary walk right, the vector space wrong. Special Tonality is created when the barycenter of a single algebraic sub lattice is fixed at concert pitch, so that the tonality seems to be a function of the observed musical key. This leads to a monocentric view in which polyphony is a composition where every tone seems to have but 1 point of origin since there is 1 algebraic lattice, and its center is shared with the logic sub lattice so there is no tonal redundancy and movements of pitch and position are locked in concordance. Since we can only hear 1 fundamental tone, we might be tempted to conclude there can be only 1 tonal center in all music, and think of tone movement in a simplified way since the affine transformation of musical keys results in co-incident (parallel) transposition of musical keys that is merely a text-shift code in a 12-letter alphabet. But on guitar keys are not co-incident, just similar and equal.

Section 6.6 Sound Horizon

We hear sound on a sound horizon that is the line of tonality which embeds pitch in our brain, so that it seems pitch can only rise and fall. But if you believe that tonal movements in higher dimension are not aesthetically and technical important then you are precisely the same as a flatlander who couldn't understand when the sphere came through town.

Section 6.7 Barycentricity Theorem Is Magical Illusion

Nearly 300 years later I followed Euler's binary walk from where he laid the egg (modern music theory is a donut) to the right manifold: it isn't pitch value space at all because pitch value space is undefined.

Music is always and only a pitch-position space defined by the cipher between pitch and position, of which there can be only 1 such vector per system.

This contradicts the teaching that in affine transformations the fundamental tone is “forgotten” so that tonality is what is left after the tonal center is removed: a curious idea at best.

Music is a tone value space (not a pitch space) with 1 tone center and at most 1 cipher connecting pitch and position. The 1 cipher, the line of tonal project is precisely tonality. [Barycentricity Theorem] Special Tonality is the best-known form in music, foundation of the common tonal period, a world where transposition seems to be a just mathematic function (as opposed to the algorithm it really is) that you can do on the fly.

Section 6.8 Bijective Ciphers and K3, 3 Subsets

The unique property of Special Tonality is the bijective cipher. A bijective cipher means that tea cups (pitch values set) and saucers (position value set) match and stay matched. This reduces originality/expressivity of keys, but simplifies the rules of transposition. Bijection allows the Special Tonality Cipher (to a barycenter at concert pitch 440 Hz) to be regarded as if it were in fact co-planar with pitch and position axis.

But in fact the pitch-position line is not co-planar (12-arity gives k3, 3 subsets, called utility graph problem). WE there is at most a single point on the line of tonality that lies within the plane of the origin.

Tablature music writing is Boolean arithmetic and cipher geometry. The literature is replete with analysis of the cultural value of pitch vectors. I am the first person to understand position vectors in music and how they determine the pair (tonality, harmony).

Section 6.9 Barycenters in Music Notation

Special Tonality gives the appearance that tonal movement is merely 2-fold (chromatic, harmonic). The 2-fold path is actually an artifact since concert pitch is fixed. However if concert pitch changed, music staff positions would cipher, but pitch would not.

This ciphering also happens on with tablature and on the guitar itself. The barycenter of the musical staff is concert pitch, and in tablature is the lowest note on the lowest string. The tonal center of the musical key is defined by the musical staff but the tonal center of the guitar is barycentrically defined relative to the lowest note. The guitar is not simultaneously tune to both E2 and concert pitch.

Therefore music notations are tone positions defined independent of barycenter. Whe barycenter changes, both position and pitch notation cipher out.

Section 6.8 Expressivity of Special Tonality

The characteristic of Special Tonality is that musical keys are nearly co-incident but a small amount of cipher discontinuity is created because the diatonic tuning rule partitions the scale 5/7 and not 6/6 (for reasons well-known in music theory). The positional variance on piano is minimal and concerns only the asymmetry of whether notes fall on black and white keys. This makes each piano (or diatonic key) a little different in tonality from every other key. When there are no sharps or flats, and there is no tuning rule so every note is the same, the musical keys are truly co-incident. Exactly co-incident keys are boring and this makes the cipher less original and less expressive.# The problem with the musical donut theory is that the donut allows for at most 2 centers when in fact at least 2 centers are required. A torus does not allow the pair (pitch, position) to move by pitch at all since concert pitch is fixed. In Special Tonality tones move either by position or by pitch-position together as a composite. If the line of tonality defined by the cipher is not co-planar it simply cannot also be co-toroidal.

Chapter 7 TONALITY OF A SINGLE STRING

Section 7.1 Tonality of Single String

This section repeats the formulation of a music topos for a single string before moving to the union of tuned strings.

To illustrate the model in a simple way I discuss the pitch-position relation for a single vibrating string. This demonstrates arithmetic closure and shows how the cipher defines musical tones by the pitch-position line of tonality. The fundamental vibration of a string is a function of string mass, tension, and length. The tonality of the string is the totality of expression that includes every function, relation, and element possible for the string which the cipher can form by adding to the center. A string can have only 1 fundamental but a secondary center forms the key. The string can play in any key, but not all keys are equally good. The tonality of the string is therefore synonymous with the fundamental because if the fundamental is known the system is also completely known.

Musical key on a single string are rankable by how in tune the keys are, and also by how playable the keys are.

Chapter 8 THEORY OF GENERAL TONALITY

Section 8.1 Tonality with a 3rd Degree of Freedom in Tonal Movements

General Tonality is defined by adding a 3rd degree of freedom in tone movement by allowing concert pitch to move in tempered steps. When concert pitch is not fixed as the system's designated barycenter, concert pitch is not the lowest note (true system fundamental) but concert pitch is the effective barycenter, the note by which all other notes in the system are defined. Of course, instruments can tune to any pitch value and still be at concert pitch in a tempered scale, but this merely reflects the principle of a single system fundamental, but the issue in general tonality is symmetry and degrees of freedom.

A Theory of General Tonality assumes that concert pitch values can move with the same scale metric as tone value space. We always assume the musical sequence of the scale used by in polyphonic union is common to every voice and metricizes tone value space.

Section 8.2 Real Closed Field in Music

The real closed field is another pair (0, 1) and there is also the (0, 1) of the logic sub lattice of the scale, tempered or no. The line of tonality defined by the cipher is precisely the pitch-position graph formed by the point (1, 1) which defines the real closed field that gives tone value space a tempered metric. That's the same line that connects sub lattices. The line of tonality is only a theoretical (since the notes of the overtone series do not fall on the line and in fact based only on the single point (1, 1). The line is a point. (Duality of Line and Point Principle)

Section 8.3 Reasoning Out Multicentricity: Primary, Secondary, and Tertiary Centers

Music can have many tonal centers but there can be at most only 1 system fundamental. Therefore all centers must be at the same point, which is the system fundamental.

By theorem, there can be at most 1 point used to define the cipher (besides the origin) since the harmonic functions converge on a most 1 point (Convergence Theorem, Gemignani 1972). Furthermore, a string can have only one fundamental, a tone value vector field can have only 1 point of origin, and a scale can have only 1 first note. Also spheres and equilateral triangles can have only one center but rectangles and cubes have many centers. A donut have two center that are not the same; how are more centers be added to the donut? Is 1 the real center? It is possible for music to have many centers so long as they share a common point. [Assertion 2: Monocentricity Theorem: A musical system M can have multiple centers but they must be coincident with the fundamental.] As I show here, that there is more than 1 " key" center in a music system such as guitar. Any instrument with more than 1 string is correctly understood as multicentric.

Section 8.4 Concert Pitch and the Spiral Wedge

If concert pitch is allowed to rise continuously as a monotonic function, the result is that the line of tonality forms a spiral wedge as the (1, 1) points winds around the pitch axis. (Chew 2000 Ph. D Thesis Claims spiral wedge and torus can be superimposed but these manifolds are not homeomorphic.). Subject to algebraic closure by the octave interval the spiral wedge becomes a sphere.

Section 8.5 Embedding and Forgetting in Affine Transformations

The fundamental tone cannot be forgotten by an affine transformation without making the key undefined if the R: Z connection is cut. The musical cipher defines tonality as the projected image of the harmonic message embedded by the cipher tonality vector in a musical system. A cipher is a zero crypto logically because encoding and encoding vectors cancel out information in order to conceal a secret; there is a kind of universal cancelling out created by orthogonality in that any 2 tone vectors multiply to 0, a property called orthogonality (like a crossword puzzle). The cipher is an affine transformation that hides the fundamental as the real point of origin, called "forgetting", or substitutes, called scrambling, which is the basis of a cesarean code language.

By either text substitution or text shift, or both, the code is formed by an addition operator of the tonality cipher vector and its algebraic closure operators defined by 2 or more center points. Cipher here refers specifically to a vector that is a difference function of 2 algorithms, used to change musical states of system between the 2 intonation algorithms. By using ciphers, I discovered the principle of tonal duality by noting that the harmonic message is embedded in a music system like a code. Defining the Line of Tonality by the Tarski-Seidenberg Theorem

Embedding concepts (Poncelet) led me to define tonality as a line of projection instead of the traditional view of tonality as a point, using Desargues Theorem to guide the understanding of projectional invariance that underlies commuting algebras of Hilbert Spaces.

The line of tonality is the line of affine projection between the fundamental tone in a musical system and the tone center which embeds the harmonic image as a code defined by the algebraic closure operator of the projection (Stone 1937). I show here that the line of tonal projection that embeds the harmonic image in tone value space topology is precisely the affine projection between the logical and algebraic sub lattices of the complete musical lattice (Tarski-Vaught Test).

Section 8.6 Semi-algebra Is a Coherent Theory of Tonality

A general, coherent cipher theory of dual tonality states the cipher vector is the tonality determinant because there are at least 2 tonal centers in any defined musical system so there must be a cipher between them. Multiple centers in music are important since a single tone logic sub lattice can be related to multiple tone algebraic lattices each with their own secondary center partitioned by an intonation algorithm that defines cipher tonality as the logic-algebraic interface. The intonation vector individually connects the sub lattice points of origin so that the algebraic sub lattice is partitioned by separating the prime and secondary ideals, so that the tonality becomes defined by the multi-centric vector projection and not the observed musical key.

Section 8.7 Grand Illusion of Tarski Tonality

Because the algebraic and logic sub lattices share the same scale metric it may seem that General Tonality is not important aesthetically and technically, because in the tone value space metricized by the scale sequence conceals a text-shift/substitution code, and we are misled by our perception that the observed key the is different from the true key, any Special Tonality tells us the fundamental has been lost by transposition. Because pitch values are projectional invariants but position values are not, the tonal projection profoundly affects tonality but the observer, hearing the same notes and observed musical key, cannot easily perceive why the " equal but not co-incident" arrangements differ in expressivity since pitch is ciphered out. Tunings combine both text-shift and substitute codes and therefore more difficult to decode than simple key change. Text substitutions are not transparent like text-shift codes because the cipher results are equal but not co-incident. When we assume that Special Tonality is not important in compositions and the result is that multicentric tonality in music is not properly transcribed and understood. This problem is important on guitar where it is impossible to play or understand music properly without finding the tuning.

Chapter 9 MATHEMATICAL FORMULATION

I attempt a more rigorous mathematical formulation here as a topos to explain the ordered Abelians by the semi-algebraic sets.

Section 9.1 Binary Functions

Music is a binary construct composed by the finite union of notes and intervals in a binary path of steps and vertices with order 2. Any music system can be constructed by adding intervals and notes to the fundamental, and any tone value can be change to another tone value by adding the correct value. There is a well-known musical duality of note and interval such that whatever is true notes must also be true for interval. The same duality is found in pitch and position values, algebra and logic, syntax and semantics, harmony and melody, originality and expressivity which are crypto logically independent. Tonal duality is precisely the duality of line and point in geometry.

Section 9.2 Semi-algebraic Topos Formulation is a Point-Set Topology

I assert here that the musical topos makes it possible to trace the duality of the pitch-position line back to its point back to its point of origin, provided that the Boolean Arithmetic and its field of prime ideals is sufficiently unique so the partition function of the cipher, that is the binary-algebraic interface can be determined. To formulate these ideas I start with a general expression for the topos of all possible tonality in music as a function of the fundamental tone is an ordered orthonormal triplet:

M(fundamental) = (fundamental, {scale}, {tonal centers}

Next I formalized this intuitive formulation by defining (as a function of the fundamental tone x) the tonal algebraic functions set as {f(x)} where f(x) is the general of the cipher as a line is Y=X+B, the scale is partially-ordered K12 ring called a 12-arity function for the scale, and the tonal algebraic-logic interface is the logic functions {g(x)} where g(x)>0 defines a set of tonal centers secondary to the fundamental which may be a musical key or strings of the guitar. Therefore f(x) is the algebraic cipher and g(x) is the logic cipher. The fundamental and the secondary tone centers are the partition defined by the summation vector whose g(x) values according the tuning algorithm vector, which is the projection defined by the summation vector (same as guitar tuning). The functions sets (i.e., algorithms) {f(x)} and {g(x)} are precisely the semi-algebraic functions described by van den Dries (1998). In addition van den Dries uses the term semi linear since f(x) is always first order. This proves that M is a form of Boolean Arithmetic that is precisely the same as it field of prime ideals that is precisely the line of tonality between logic and algebraic sub lattices that is precisely the affine transformation between a pitch and position values that defines tonality.

Section 9.3 First-order Arithmetic Language

The Boolean Arithmetic Language L I recovered by systematic study of guitar Tone Arithmetic L={+,>,0,1} under a 12-arity sequence function in which the real closed field is (0,1) is defined by 1 log cycle in frequency is the universal metric. Tempering is not required for complete lattice structure. Formalizing the music topos we have the general expression for tonality as a function of fundamental x, M=(x,{f(x)},{g(x)})={+,>,0,1}| 12-arity (1)

Section 9.4 Metric and Group

The orthonormal group is torsion free so that affine projections in tone value space are non-rotation. The vectors space of the Boolean Arithmetic is orthonormal ($a = a^2$) which is due to the scale metric and not tempering. Tempering causes the algebraic and logic sub lattices to conform, but the scale and point of origin create and orthogonal logic field. The pitch-position line (which is the line of tonality between (0, 0) and (1, 1) is not co-planar (K3, 3 subsets or utility graph problem). The pitch-position graph therefore cannot be drawn correctly on paper, but can be shown by Hasse diagrams of partially-order sets.

Section 9.5 Topologic Manifold

If concert pitch is not fixed, but instead increases continuously, the pitch-position line rotates about the pitch axis to form a spiral wedge which is converted by algebraic closure to 1 log cycle in the form of a sphere. The music topos cannot have a toroidal manifold since it can only have one fundamental, so the pitch-position line is on the surface of a sphere. The sphere is also the required manifold when more than 2 centers exist. This is a peculiar paradox since the line of tonality is formed by a single point, but this must be so since harmonic functions converge on at most 1 limit. It follows then that the multiple centers of the music topos M must share one center which is the point of origin in the logic sub lattice which must be the true tonal center equivalent to the fundamental and not the center of the observed musical key.

Section 9.6 Partitioning the Scale Sequence by Tone Centers

The function set {g(x)} is a partition of M called the tuning rule that separates M into prime ideals, which are tonal centers secondary to the fundamental. The relation between g(x) and x is defined by the tuning vector T={Ci} for all i and the prime ideals are given by a summation vector Σv ={ΣCi}. So Standard Tuning has a tuning vector (0 5 5 5 4 5) and summation vector (0 5 10 15 19 24). Parameters in these vectors are tonal intervals between the fundamental and the prime ideals which completely define the musical system.

Section 9.8 Importance of Discontinuity to Expression

The musical system is completely defined but not necessarily smooth or continuous. On guitar this means that the position of notes is always correctly known but may not be fully expressive, or even playable. An important aspect of any semi-algebraic set is that partial order extends to any tone values, so that any 2 tone values of the same class are rankable.

Section 9.9 Tempered Metric and Semi-algebraic Rankability of Keys

When the scale is not tempered the non-coincident cipher vectors (metric of over-tone series is not normalized) are clearly rankable since almost all the keys are out of tune. When the scale is tempered we are less able to hear that keys are not equal but the non-co-incidence of pitch and position still creates good and bad keys: the best key is the key of the fundamental, the key of the dominant, sub-dominant, supertonic, and so on down to the worst key which is always on the leading tone. The key rank is a function of the fundamental tone. The untempered scale of the string shows clearly that the musical key of the fundament is favored as the most expressive key and the other scales are rankable for expressivity. This property of rankable tone values extends to the most complex systems possible, in which computational power is not lost but rules may be too awkward or complex for much expressivity. Rankability is a fundamental characteristic of all algebraic sets. This means that position vectors, given in this equation are rankable for expressivity in a fashion analogous to the way a composer ranks chords and scales for cultural value. When the scale on the string is tempered the algebraic and logic sub lattices conform, we do not hear the favored key as well. However it is clear that the musical keys on 1 string are still rankable by playability.

Section 9.10 Defined Pitch-Position Values

We can understand the pitch-position relation by noting that if a pitch value on the string is unknown it is not possible to find the fundamental tone, but if a pitch value and a position value for 1 note are known then the entire line of tonality is discovered, and it is a simple matter to count back to the fundamental tone. In other words given a pitch value alone is not sufficient for the fundamental to be determined, but given any pitch-position the origin is determined just by counting. There is a well-known principle in music which state that is the tonal center is not known then notes, chords, scales, keys are not defined. Pitch value by itself does not establish at position in the absence of an intonation rule. Note that any musical figure can be projected on to the string by using the arithmetic closure operator of the fundamental. This means that when a note on the string moves out-of-bounds, it is reflected back onto the string by an octave inversion.

Section 9.11 Similar and Equal But Not Co-incident Projections

Therefore when a musical figure is projected onto the string, the result is similar and equal but not co-incident. Clearly affine projections which moves notes in different direction cannot be concordant. It is remarkable in musical systems that 2 arrangements can have equality and similarity but not co-incidence. In the distinction between equality and co-incidence lies the secret of the musical cipher. On a single string Special Tonality applies since there are only 2 tonal centers possible. However it is certain that the fundamental, the scale position and the instrument position share the same center so that 1 string is not correctly thought of as donut.

Chapter 10 DISCUSSION

Box II: "... the musician must act like the architect who, worrying little about the bad judgments which the ignorant multitudes pass on the buildings, builds according to unquestionable laws based on nature, and is satisfied with the approval of the people who are enlightened in this matter." Euler's 1739 original description of The Musical Donut.

Section 10.1 Cipher Theory

The cipher theory of music explains the technical basis of a profound aesthetic illusion in which the true tonal center is hidden from observation by ear (because of a form of tonal addition we cannot hear) but yet the cipher is unmistakable by tonal geometry and arithmetic. Since the cipher zeros out the pitch (that is the ciphers adds up to zero, so that they have no effect on pitch just position), music is encoded in a very strange code language that clearly affects tonal expression but yet cannot be heard directly by listening to individual notes. I developed a dual theory of tonality because I noted anomalies during a systematic study of guitar tunings (including that the observed and actual musical keys are not the same, that tonal movements are 3-fold and that the tuning of the guitar determines tonality so that to correctly transcribe music I had to use Boolean Arithmetic to understand geometry).

Section 10.2 Guitar Tunings

The vast majority of guitar tunings are judging by popular usage, utterly useless for harmonic expressions. I mastered the method of using the guitar tuning cipher to change guitar key and tuning at least 10 years ago and I noticed that the procedure involved non-commutative algebra and then logic. I realized that in order to properly understand guitar music I had to locate the correct guitar tuning first. Once I knew the guitar tuning I could transcribe recordings accurately using standard musicianship skills but without knowledge of the tuning I was not able to find the correct position for notes even if I could accurately find the pitch values. I learned that it is very difficult to establish the rules of syntax and semantics for a guitar tuning, which requires years. Furthermore, I found that I wasted considerable time trying to learn arrangements in publications that were incorrectly transcribed. Publications cannot be assumed to depict tablature correctly if the tuning is correct.

Section 10.3 Intellectual Standards for Tablature

There seems to be an industry standard that says tablature is considered inferior to musical staff but in polycentric polyphony this is clearly not true.

Two more orthogonal pairs: (Tablature, Musical Staff).

The tuning itself is a powerful abacus-like system for calculations regarding tonal movements, and particularly for tonal movements that cannot be heard. (These occur where there are more positions than pitches so that a position can move without change in pitch, and also between tunings and keys). I realized the best way to learn a new guitar tuning is to take known harmonic figures and project them into the tuning by the tuning rules. By repeatedly transforming pitch values into position values by the tuning I came to realize that the geometric tonality on the guitar fret board is not a function of harmony.

Section 10.4 Tonality is not the same as Harmony

Another pair: (Tonality, Harmony) are not the same but in Special Tonality they are close. The most peculiar effect of the tuning is that when you play the same pitch values in different tunings, the musical expression is different. This seems to make no sense at all, but it is easily demonstrated. The explanation of this peculiar observation was stated by Desargues: The geometric configuration of points in projection is determined not by the points but by the straight lines which project them.

Section 10.5 Symmetry: Are Guitar and Piano The Same?

In the original form my model was applied to the guitar where M = (pitch value, string, fret) is used to find position values by pitch and pitch values by position. I thought at first that the equations I used were unique to guitar. Gradually, I realized these expression are the result of a general form of tonal movement in where $y = x + b$ means that any 2 orthonormal vectors can be related by adding vectors. It seemed that guitar and piano were controlled by different rules of tonality, but they use the same tempered scale. How can it be that guitar tunings sound different but use the same scale? Although I am quite familiar with the arithmetic and geometry I have until recently been at a loss to explain the origin of the cipher function. In this report I formalize a crypto logic theory of music.

Chapter 11 CONCLUSION

This study of General Tonality has two very surprising conclusions which turn modern music theory on its head.

Section 11.1 Music is A Ball Not a Donut

A circle (with a center point defined) is homeomorphic to a line, and a line and a point are the same. If there is a center point then there is a line through that point. The existence of a tone center by itself involves a tonality line. A donut is incorrect because the manifold cannot possibly have centers indifferent locations. Two centers, yes; two locations no. There can be many centers in a music system but they must be homotopic; a true monocentric music is undefined because the cipher is 0 (so pitch and position are not defined.)

Section 11.2 Pitch-Position is Not Planar

While planar projections of pitch-positions are clearly not correct, they do have computational value just like rectangular maps. Rectangular coordinates create confusion for guitar. Planar projections of pitch-positions (such as the music staff and chord charts) however fail to depict guitar music movements correctly, which explains the low intellectual standard of guitar music published today which almost always shows the correct pitch value at the wrong position. This also appears to explain why most guitarists do not read music: they use position logic which is probably more powerful and robust than pitch. Many guitar players have no concept of what pitch they play but they certainly understand guitar tonality.

Section 11.3 Music is an Algorithm Not a Function

On guitar (to 12 fret, Standard Tuning) there are 36 pitch values and 72 position values. There is therefore no function that accounts for the pitch-position relation.

The recognition that the pitch-position relation is a true algorithm (as described by Tarski for any complete lattice) is critical for guitar since it forms the basis of a first-order illusion in which the listener is unable to determine the true tonal center (the guitar key) that the composer used to create music.

In the special case of tonality limited to concert pitch tonality, the musical key (and therefore tonality) seem to be functions, but functions by themselves do not have expressivity or originality, and they cannot have cultural significance. Therefore understanding algorithm is vital to understanding tonality.

It is interesting to note that until recently algorithms were not patentable (need citation).

The tonal line connecting to concert pitch is commonly abandoned completely by guitarists. Given guitar tuning, retuning changes are easily understood, but in combination capos, retuning, key changes, and tuning changes create a daunting tonal redundancy that makes guitar far more original an expressive than piano.

Clearly ciphers have great value in music. Algorithms have cultural value because they save us time, and also by virtue of their harmonic expressivity and originality, which in turn defines cultural significance. Ciphers save time with rules for writing music correctly and because they avoid right pitch-wrong position gibberish so common in music (and music topology) published today.

Section11.4 Tonal Redundancy

The moderate tonal redundancy within the guitar tuning and the enormous tonal position redundancy between tunings is a difficult problem for guitar because as symbol redundancy increases rules of syntax and semantics become overwhelmingly complex, and therefore take longer to learn.

Understanding the pitch-position correctly allows the guitarist to efficiently write and understand music which might otherwise seem impossible to understand. The topos I describe here applies to all music and can be formulated in a number of useful ways since f(x) and g(x) can be any semi-algebraic tone properties: (pitch, position), (string, fret), (melody, harmony), (algebra, logic), (syntax, semantics), (originality, expressivity) and so on.

My description of the line of tonality created by the music cipher is used in my tablature search engine to define the path by which culturally-significant tuning-keys are discovered.

In conclusion, music can have many centers but all the centers must be located at the same point as the system fundamental. This is possible only if the pitch-position triangle is equilateral and the manifold is spherical. It cannot be that music has more than 1 true center because then multiple homotopic cycles would be impossible in polyphony.

At every level in the music topology there is a requirement for 1 and only fundamental center.

Music is composed like a code by a cipher that adds to a single point to make a composition that is the finite union of notes and intervals. The true center can be hidden but not lost.

Box II Heebeegeebee Tuning

An observation about the cultural value of the cipher: I state here my claim that the word Heebeegeebies was not coined in the famous comic strip drawn by a white man around 1910. This word in fact comes from the well-known negro guitar tuning Open E Minor or EBEGBE, which is at least 100 years older than "Pogo."

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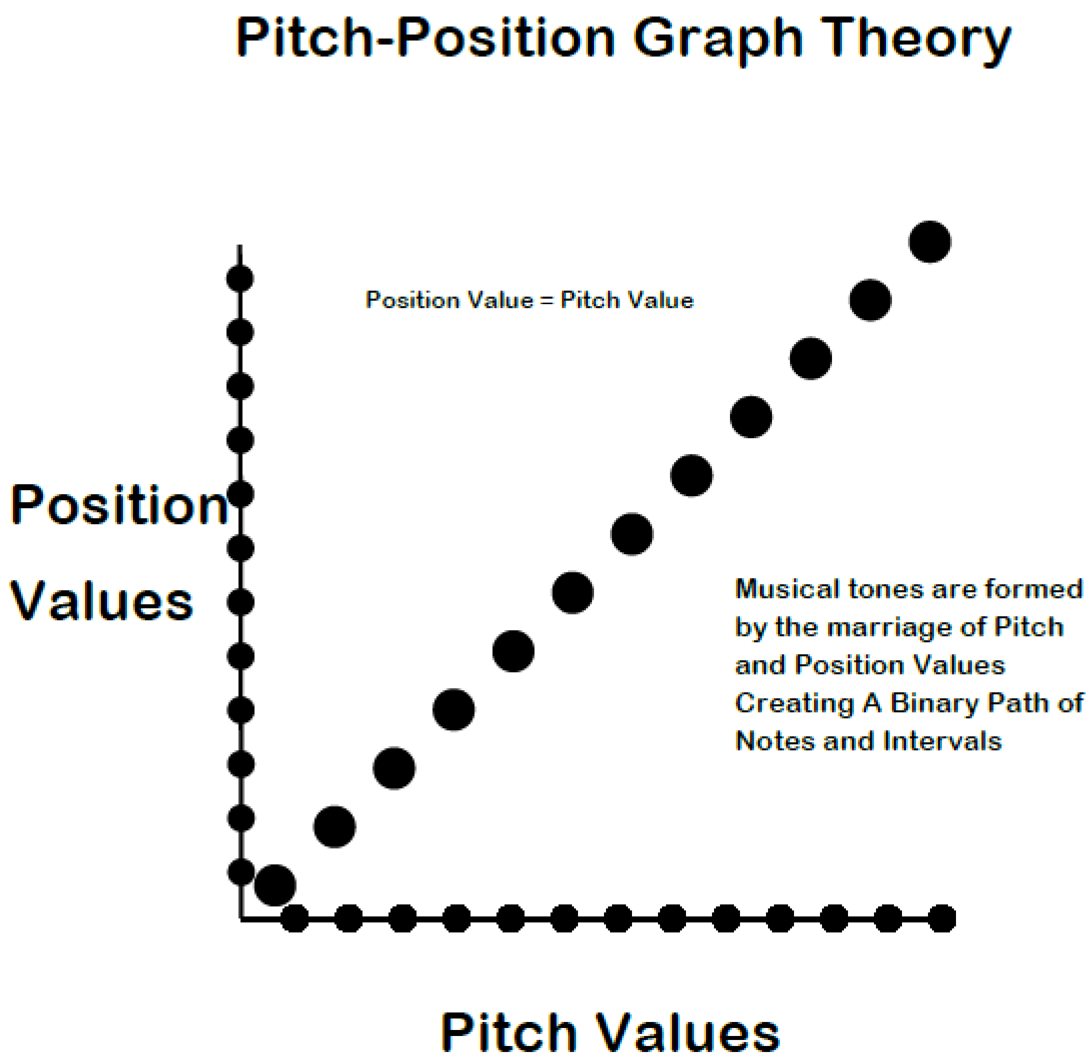


FIGURE 1 General Relation of Pitch and Position

The General Pitch-Position Relation is a line where pitch value = position value. The path of tonal movement is binary because it consists of points and intervals. The points are musical tones formed by the marriage of pitch and position.

Pitch-Position Graph Theory

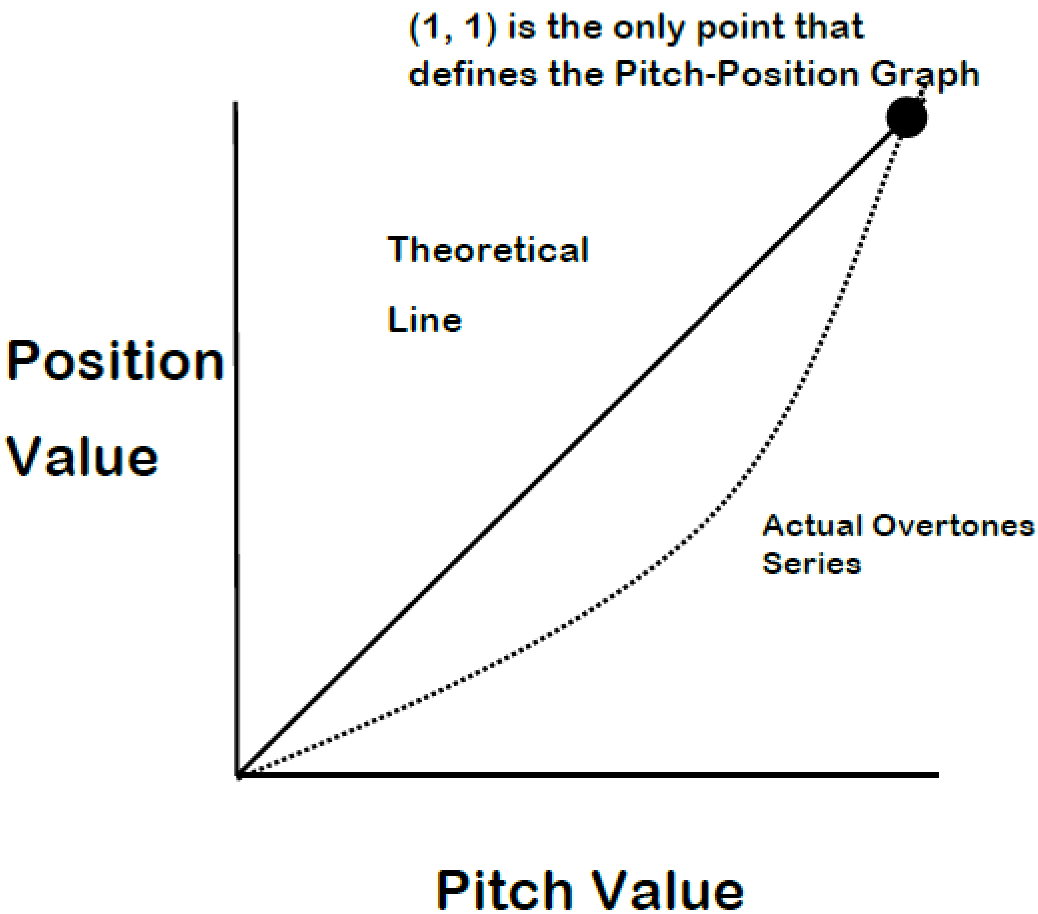


FIGURE 2 Constructing The Pitch-Position Triangle

The Pitch-Position Line is Formed by a Single Point of Convergence at (1, 1) Defined By Pitch = Log Frequency. The point (1, 1) by 1 log cycle in frequency = 1 octave in pitch values as well as 1 octave in position values. This is based on a mathematic principle that functions converge on at most 1 point.

Pitch-Position Graph Theory

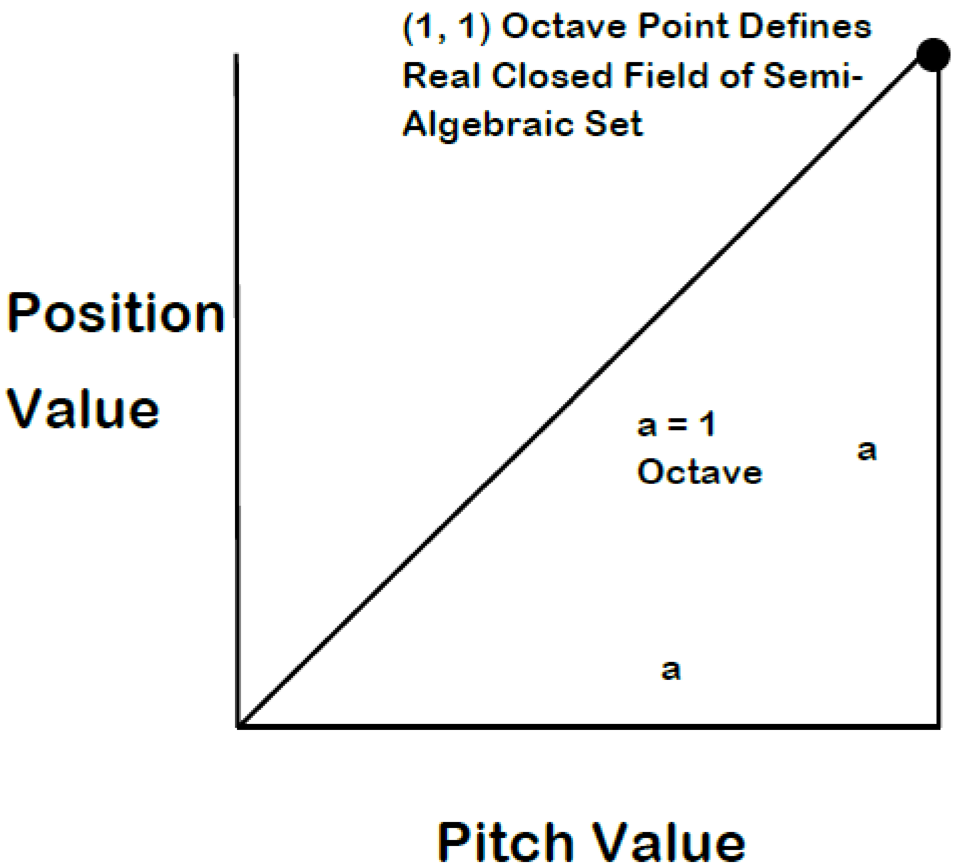


FIGURE 3 Pitch-Position Triangle is an Equilateral Right Triangle where each side is 1 Octave in length so the Pythagorean formula $a^2 = b^2 + c^2$ for the sides of a triangle is reduced to $a = a^2$ which establishes orthonormality. Note as drawn there is only 1 right triangle.

Pitch-Position Graph Theory

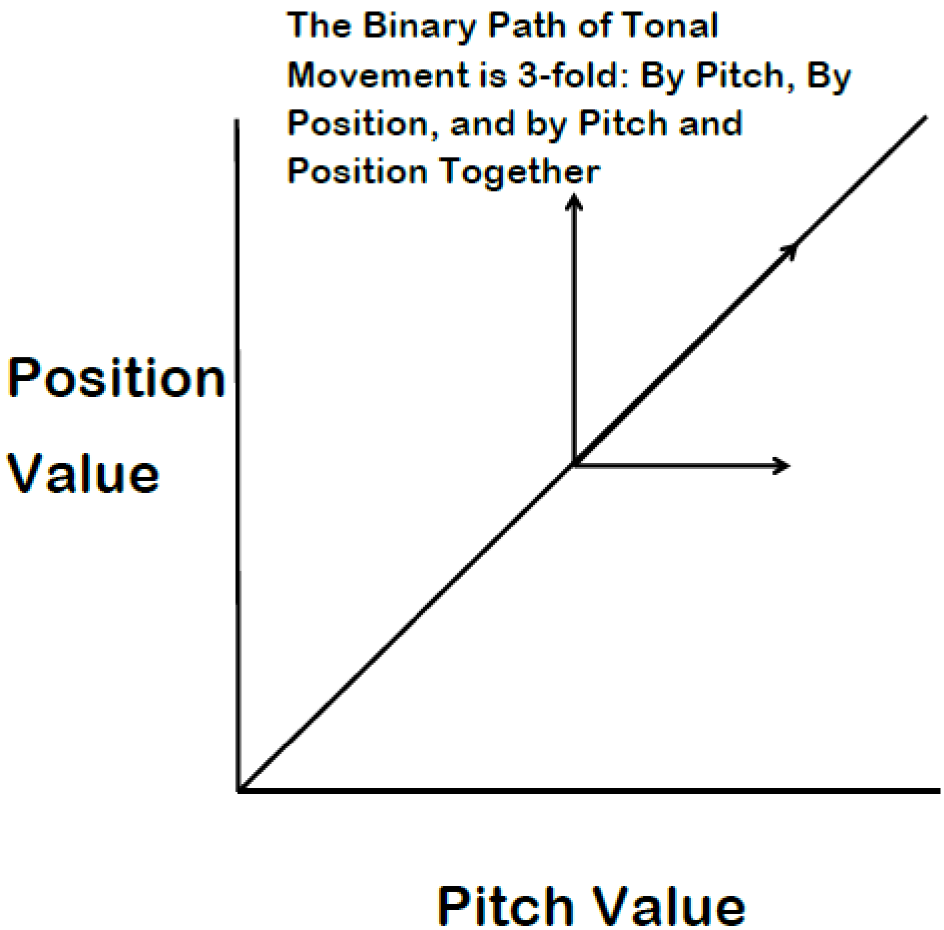
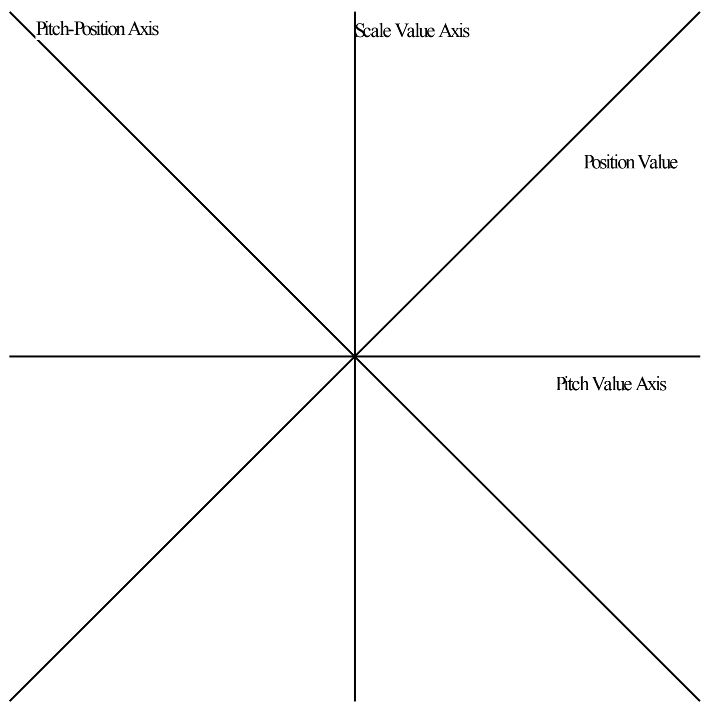
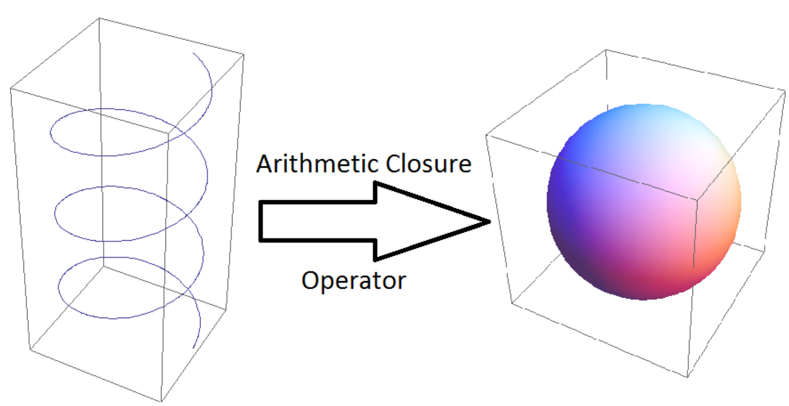


FIGURE 6 The Binary Path of Tonal Movement is 3-fold since there are 2 directions that create a new pitch-position relation algebraically and then a third path in which both pitch and position change as a composite function in which there is tonal movement but no change in pitch-position relation.

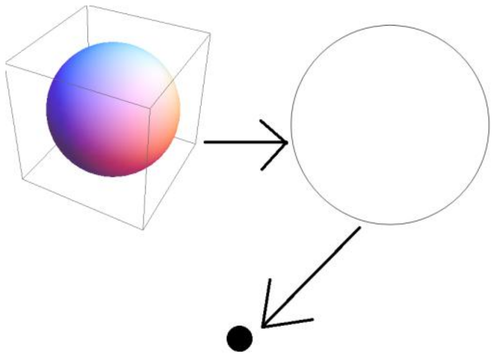
Pitch Value-Scale Value-Instrument Value Graph (Triple Function)



Open and Close Manifold

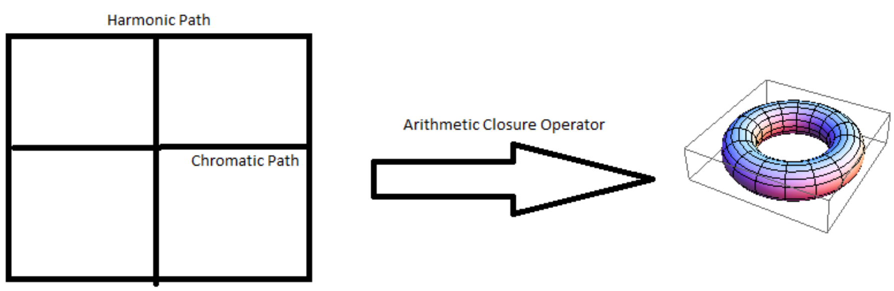


Homeomorphic Pitch-Position Manifold



First, S^3, a sphere with 3 great circles = 3-fold path SO^3

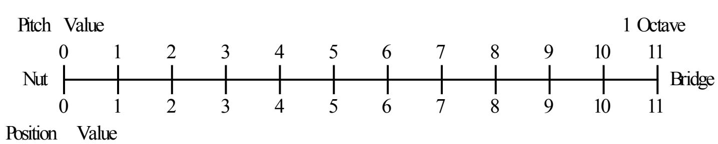
Second, S^2circle, chromatic harmonic (5 th or 4 th) Scale Value,Instrument Values, Pitch Values, Third, Point, (Musical Tone)



Torus is homeomorphic to a plane, Torus has 2 centers which are not the same.

Pitch-Position Relation on Single String

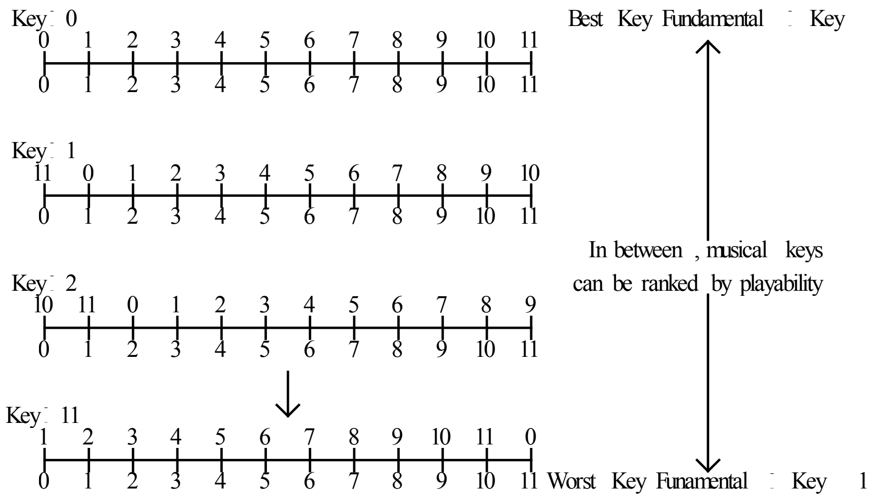
Pitch Value = Position



The Barycenter is 0 pitch and 0 position.

This characterizes string in any tuning.

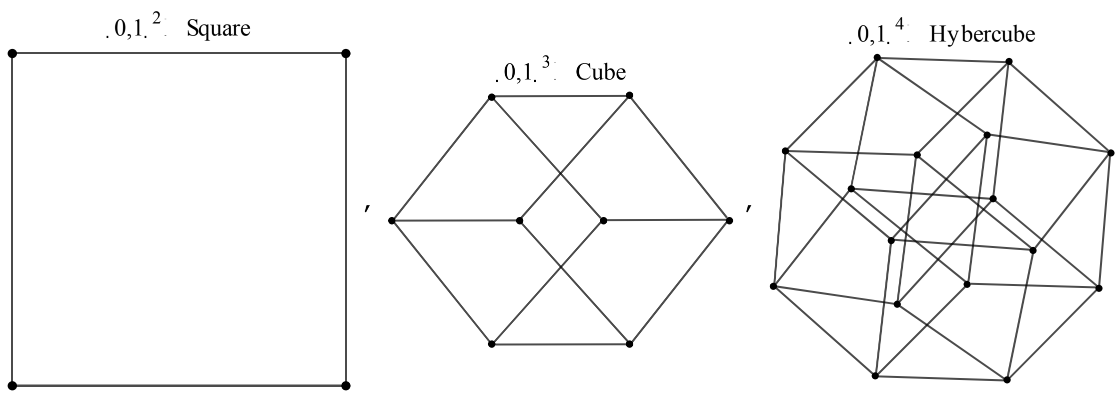
Scale Values



Ranking Tempered and Untempered Key

Tempering the scale makes a metric space where algebraic and logic lattices conform.

Logic lattices are always orthonormal modules:

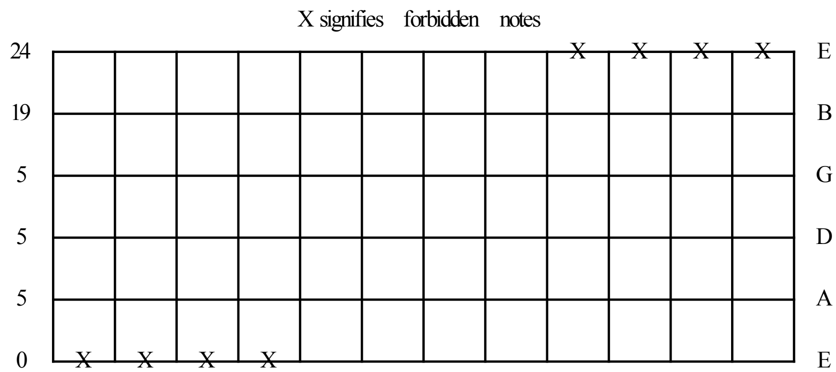


All scales are logically tempered. Untempered keys are highly ranked because most keys sound out of tune. Ranability is present in all semi-algebraic sets.

Tonal Redundancy I

6x12 Guitar Fret Board at 55545

6 String x 12 Fret = 72 positions



Barycenter
$$PV_0 = E2 = (1{,}0)$$
$$E3 = (3{,}2)$$
$$E4 = (6{,}0)$$
$$E5 = (6{,}12)$$
3 Octaves = 36 notes

36 Pitch Value $\neq$72 Position value

Therefore a paradox, pitch = position

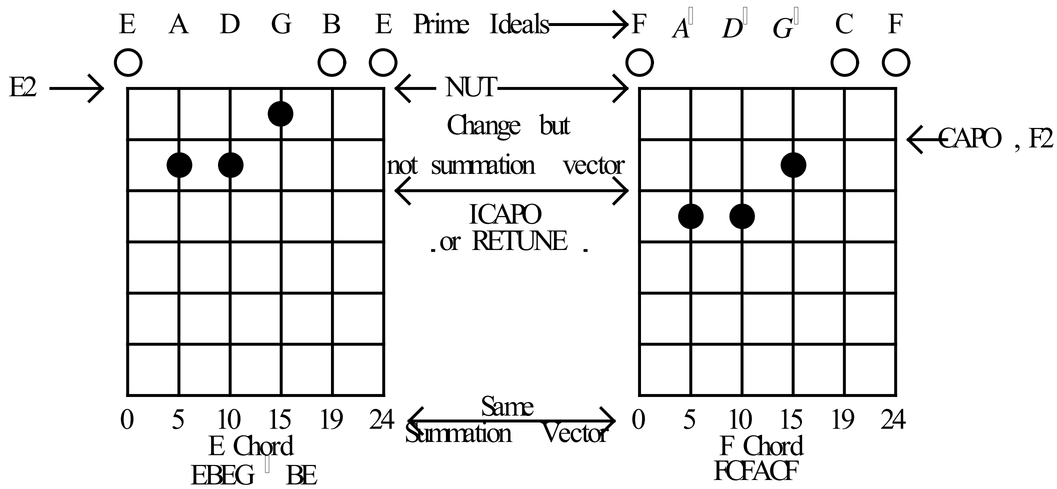
but 36 pitch values $\neq$72 position values

No function relates pitch to position

Note: Forbidden notes $N = 5^2 + 5^2 = 50$ are intervals on strings 1+6 which respectively cannot be played toegether since they are on the same string.

Tonal Redundancy II

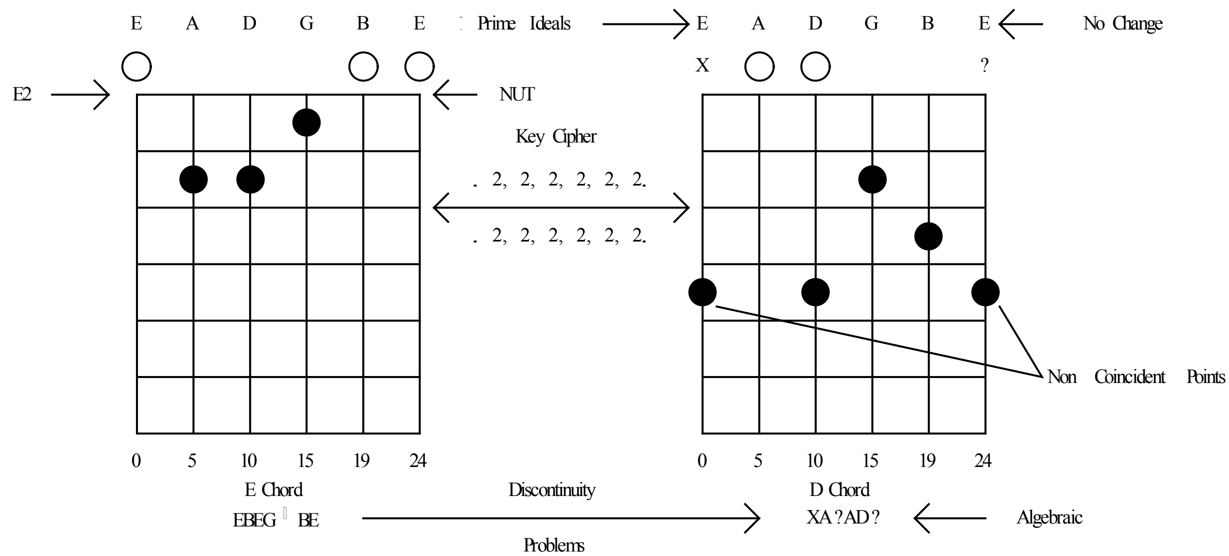
Redundancy results by similar and equal but not co-incident affine transformation



Original and transformed chords are co-incident because pitch-position relation has not changed. Tonality is altered in only one dimension which is a change in register. This is tone movement direction number 1: pitch-position move together.

Tonal Redundancy III

Non-coincident movement



$$Algebra + Logic \rightarrow \begin{array}{c} F^{\#}ADADF^{\#} \quad Rankable \\ F^{\#}AF^{\#}ABF^{\#} \quad Choices \\ \uparrow Chord\ voices\ changed \\ Sounds\ different \\ D\ Chord \end{array}$$

An E Chord is transposed

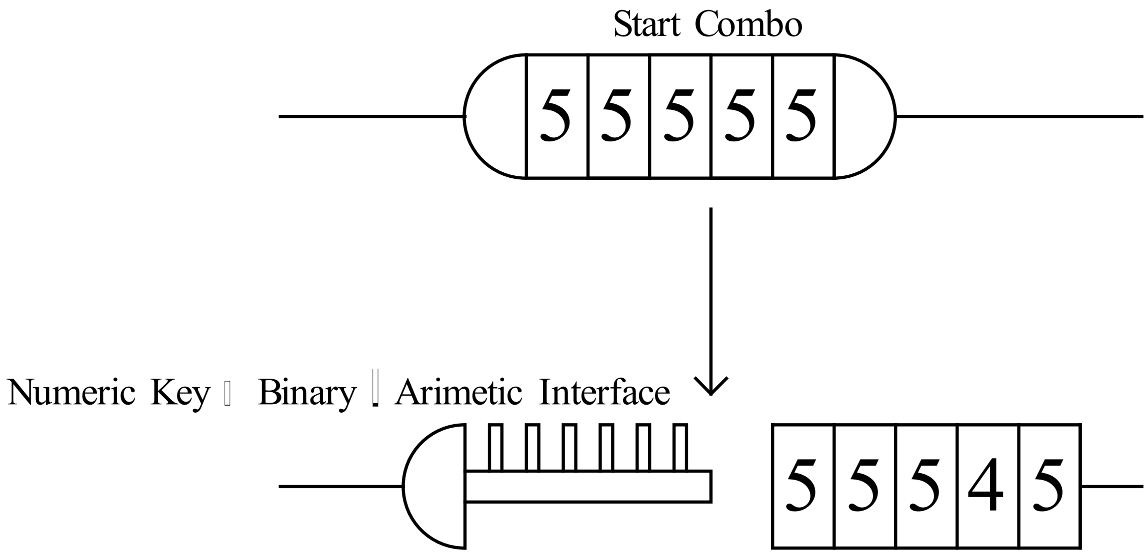
Note <u>Non Coincidence</u>

1. Note on string 1 is <u>out of bound</u>
2. 2 notes fall on string 2 – which is correct?
3. Note on string 6 moved to string 5
4. Another note in chord replaces notes on string 1 and 6 by harmonic replacement

This is tone movement direction number 2, pitch moves in 1 direction, positions another.

Numeric key as a combination lock

Closed Position = Can’t understand tabs, nonsense



Open position = Tabs make sense

Search Engine: try every combo by brute force or can be guided by expressivity ranking.

Appendix II Brief Theoretical Formulation

Music is a re-iterative composition by the finite union of notes and open intervals added to the fundamental tone, which is created through a pitch-position intonation algorithm in which every possible tonal value function, relation, and element in the system is determined by the affine projection of the R:Z vector connecting the monotonic function of pitch to the barycenter of the tone values vector stack. To make this concept more rigorous let us define the fundamental x and function sets {f(x)} as open intervals and {g(x)} as notes which are secondary centers called keys, that act as additional centers upon which new tonal constructs can be built in a binary walk of notes and intervals. The musical topos M =(x, {f(x)}, {g(x)}) a graph of functions f(x) which is a tone value triangle (line) in the form TV(1) + TV(2) + C = 0 which is called a cipher because the value of the linear function is 0, while g(x) > 0 is a greater than function that makes the scale sequence a partially ordered set in which each note is greater than the secondary center, just as each secondary center note is itself formed by adding to the fundamental. Since F(x) and g(x) are van den Dries semi-algebraic functions, the topology of musical topos M is completely determined. Perhaps the most characteristic aesthetic in music is the orthonormality of the Hilbert space created by the tone values vector stack resulting from the metric of independent scale sequences, which is extra-ordinary because any 2 tonal values vectors (not the same) are always equal and independent so that every angle is perpendicular and every distance is 1. Since every tone value element satisfies a = a2, it follows that the Pythagorean distance formula becomes is a + b = c and that 1 + 1 = 1 is the equation for both a pitch-position unit triangle, equilateral triangle with sides of 1, and a unit sphere with radius 1 where 1 is defined by 1 log cycle. The tone values vector space is highly ordered but still undefined without a connection between the barycenter of the tone values space and the monotonic function of pitch by an R:Z cipher vector whose properties are extended to the entire vector stack through the barycenter. This principle is well known in music: a note is undefined without a key. Therefore a musical key is undefined without a fundamental. Trying to define a key in the absence of a fundamental is analogous to trying to read time by the minute hand of the clock, with the hour hand or even the same center as the hour hand. It cannot be however that there are two sequences, chromatic and harmonic, in the same key but with different centers (that is without the 2 paths sharing the same first notes in the sequence). In the Hilbert Space tonality is expressed by tonal movements that are not perceived directly on the sound horizon where we hear the harmonic image as formed by addition to the secondary tonal center of the musical key when the tonality of the music is actually a function of the fundamental. A musical system can have only 1 fundamental tone, which is not defined by a pitch value as commonly thought, nor by a position value (string 1, fret 0) but instead by the union of a pitch value to a position. The union of pitch and position is a marriage by an intonation algorithm in where the mass, tension, and length of a vibrating string (or other oscillator parameters) are adjusted to connect pitch and position by an R:Z cipher function according the tuning rules. In this way the pitch-position line is precisely the tone point. (Line-point duality principle in geometry.) This model for music topos M then describes a curious duality illusion in which the tonality of music is the composite function f x g but the harmonic message is embedded as an image in tone values space by the line of tonal projection which is precisely the vector between the logic and algebraic sub lattices of the music topos M. Therefore tonality is the algebraic closure operator of the pitch-position algorithm and harmony is the image if the harmonic position encode by the pitch-position algorithm as an independent orthonormal pair (tonality, harmony). The first-order illusion is then created where by the listener cannot understand the path of tone movement in higher dimensions,

but the guitarist can see and understand an exquisite geometry in higher dimension than are commonly recognized. This is a versatile model for music since the tonal vector pair ({f(x)}, {g(x)}) can be (pitch, position), (string, fret), (string, fundamental), (algebra, logic), (harmony, melody), (syntax, semantics), (continuous, discrete), (originality, expressivity),
and a number of other independent pairs. The sum total of all these orthogonal vector pairs determines the cultural significance of music. Since the binary path of tonal movement in music is by function f, by function g, and by f x g in composition, I have now proven that the symmetry of tonality is SO3 and not planar as commonly thought. Furthermore pitch value space without an R:Z cipher is not defined and cannot possibly be a toroidal manifold (which implies 2 fundamentals when there can be at most 1). The real problem with the musical torus is that it does not allow additional tonal centers to be added to the system to form a multi-centric tuning. I therefore assert here that to be defined a musical system must have at least 2 centers, but at most 1 fundamental.

Appendix III

The Binary Path of Tonal Movement is 3-Fold: Experimental Demonstration

By Dr. Terry Allen

Abstract: Recently I showed theoretically that tonal movement is 3-fold on a spherical basis as the composite function of line and point. I here demonstrate the 3-fold path by counting tonal movements on commonly used instruments.

Demonstrating Fold Path of Tonal Movement
On Piano:

1. The chromatic path by note
2. The harmonic path by key
3. And on the transposing piano by mechanical lever (algebraic operator) that moves the instrument position relative to strings.

On the Music Staff

1. By Position on Staff
2. By Clef
3. By Concert Pitch

On Guitar

1. By Strings so that String Number and Pitch Value Rise Together (by tuning)
2. By Frets so that Fret Number and Pitch Value rise together
3. By String and Fret as a composite function where String Number rise or fall but pitch values do not (iso-pitch line ciphers pitch not position)

    A String

1. Mass
2. Tension
3. Length

    Tablature

1. By Lower Note
2. By Tuning
3. By Capo

By Transposition

1. To change position literally
2. Also to change pitch
3. Pitch-Position can move as a composite function.

This should serve to open the mine of the spherical manifold on which music is composed.
We hear music on a sound horizon so that it seems that pitch can only rise or fall.
In this sense I am introducing the idea that guitar chords and scales are higher dimensional structures, so that those who believe music is planar resemble the flatlander when the sphere came through town and tried to explain there was another dimension.

Except in this case we simply have to look to see that the binary path of tonal movement really is 3-fold.

Appendix IV Mathematic Theorems Used in This Study

Principle	Application	
Gödel's Incompleteness Theorem	Incompleteness	
Compactness Theorem	Completeness Test Defines Compactness	
Lowehheim-Skolem Theorem	Defines Elementary Sub-model as countable	
Morley's Categoricity Theorem	Cyclic Abelians on ℤ are countable	
Tarski-Vaught Test	Decidability of Sub-modal	
Quantifier Elimantion	The Structure (ℕ, + , *, <)	
Hilbert's 10^{th} Problem	Solvable Diaphantine Equations	
Torsion Free Abelian	Countable Structure	
Presburger Arithmetic	{ℕ,-,+,<,0,1}	
Hilbert's Basis Theorem	Every Algebraic Set has finite number of polynomials	
Curve selection	Equivalance Relations are easily definable in Real Closed Fields	
Uniform Bounding	Semi-algebraic means a finite disjoint union of cells.	
Cell Decomposition	Semi-algebraic means finitely many pairwise disjoint cells C1, . . .Cn Such that X = C1 ∪….∪ Cn	If X is semi-algebraic then then is a tuning vector algorithm; Partitions Topology
Desargues' Theorem	Harmonic Collinearity; Perspective Triangles	
Poncelet's Theorem	Central Homology and continuity	
Mobius' Theorem	Barycentric Calculus	
Caley's Theorem	4-th dimension	
Tarski-Seidenberg Theorem	Tarski Projection Principle	
Kuratowski 3, 3	Complete Bipartite Graph	
Tutte's Theorem	Straight Line Embedding	
Link Condition Lemma	Triangle Simplification	
Hilbert 17^{th} Problem	Square of rational functions	
Monotonicity Theorem	Continuity	
Triangulation Theorem	Homeomorphic to Semilinear set	
Brouwer Fixed-point Theorem	At least 1 fixed point	
Tarjan's Strongly Connect Graph Algorithm	Finding strongly connected positions	

Uryshon’s	Disjoint Set must have function	
Classification For Compact 2 Manifold	Sphere or torus?	
Polygonal Schema	Represent Triangulation of Sphere as Square or polygon	
Whitney’s Theorem	2-Manifold embed in $\mathbb{R}^3$ or $\mathbb{R}^4$; 3-manifolds embed in $\mathbb{R}^5$ ($\mathbb{R}^6$)	
Winding Number	Octave	